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Non-permanent form solutions in the Hamiltonian formulation of surface water waves

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Abstract – Using the KAM method, we exhibit some solutions of a finite-dimensional approximation of the Zakharov Hamiltonian formulation of gravity water waves, which are spatially periodic, quasi-periodic in time, and not permanent form travelling waves. For this Hamiltonian, which is the total energy of the waves, the canonical variables are some complex quantities a_n and a_n^* ($n \in \mathbb{Z}$), which are linear combinations of the Fourier components of the free surface elevation and the velocity potential evaluated at the surface. We expose the method for the case of a system with a finite number of degrees of freedom, the Zufiria model, with only 3 modes interacting. © 2000 Éditions scientifiques et médicales Elsevier SAS

KAM theory / non-permanent waves

1. Introduction

We consider the problem of waves of the free surface of an inviscid and incompressible fluid which satisfies the Euler equation. It is well known that this problem can be formulated as an Hamiltonian system (Zakharov [1]) with canonical conjugate variables $(\eta(x), \psi(x))$. Here $\eta(x)$ is the free surface displacement above the point x of the (planar horizontal) bottom, and $\psi(x)$ is the velocity potential taken at the surface $\psi(x) = \Phi(x, \eta(x))$. There are infinitely many degrees of freedom, labeled by the parameter x.

Several important numerical and analytical studies have been performed in connection with the question of existence and stability of travelling (or permanent form) waves. They are solutions of the form $\eta(x,t) = g(x - c.t)$ for some function g and constant c. We refer in particular to the works of Levi-Civita [2], Crapper [3], Longuet-Higgins [4], McLean [5], Chen and Saffman [6], Mackay and Saffman [7], Craig and Sulem [8], Craig and Worfolk [9], Craig [10], Bridges and Dias [11] and Debiane and Kharif [12].

From now on, we restrict ourself to the case where the variable $x \in \mathbb{R}$, and the solutions η, ψ are required to satisfy periodic boundary conditions, in x (in $[0, 2\pi]$ for instance).

Of particular interest for searching the solutions of a Hamiltonian system, the Birkhoff reduction to normal forms (Birkhoff [13]) is known to be a powerful method of analysis. It even gives rise, for some cases of the water waves problem, to integrable reduced Hamiltonians as it was shown in Dyachenko and Zakharov, [14]. However in Craig and Worfolk, [9] the authors showed that, contrary to the conjecture stated in the preceding paper, the fifth order Birkhoff form of the full Zakharov Hamiltonian is no longer integrable due to the presence of resonant terms. It is well known that such terms will prevent the Birkhoff iterative process from converging (problem of small divisors); see for instance Siegel and Moser, [15].

On the other hand, the KAM theory (Kolmogorov [16], Arnold [17–19], and Moser [20]) indicates that under some conditions, a Hamiltonian system treated as a quasi-integrable one exhibits a great number of

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quasi-periodic solutions on invariant tori, provided that the integrable part of the Hamiltonian system is non-degenerate, and that the perturbation is small enough. By quasi-integrable we mean that the Hamiltonian can be written as an integrable one, with small quantities added, in a sense to be explained. We also require that some parameters (or initial conditions) verify a non-resonance condition of the Diophantine type: i.e. they must be in some Cantor set.

In this paper we use the KAM method to prove the existence of a solution of a simplified, non-integrable model of surface gravity waves in deep water, introduced by Zufiria [21]. As usual in KAM theory, we build a canonical transformation that takes the initial Hamiltonian into an integrable one, at least locally in the phase space. In this way it is possible to write a trajectory of the initial Hamiltonian system for each initial non-resonant condition. It is easy to show, by inspection, that such trajectories are not travelling waves. It will be clear that the method can be used for any other truncated (finite-dimensional) form of the Zakharov Hamiltonian.

This paper is organized as follows: in section 2, we describe the Zakharov Hamiltonian for the gravity water waves and we write a truncated approximation of it in action-angle coordinates. As mentionned above, we exemplify the technics on the Zufiria 3-modes interaction model.

Section 3 is devoted to the description of the KAM strategy in the general case. We identify a part of the Hamiltonian which is small and which thus can be removed by a canonical transformation. The remaining part of the Hamiltonian has a trivial trajectory. This method builds the unknown transformation by an implicit equation. This change of variables is applied to the set of non-resonant Diophantine initial conditions in the neighborhood of a point in the phase space. Then we obtain a trajectory of the original perturbed Hamiltonian, in the initial physical coordinates, by computing the inverse of this change of variables.

In section 4 we apply the KAM method to the Zufiria model [22]: we exhibit a part of the Hamiltonian which is small for a non-empty set of initial conditions. We also verify that the obtained trajectories are not travelling waves (or permanent form) solutions.

Section 5 is a brief conclusion along with some open questions and work in progress.

Let us mention that an alternative starting Hamiltonian would be any finite-order truncation of the ones given in Dyachenko [14], Craig [10] for deep water waves or Craig and Sulem, [8] for the finite depth case. We leave this task to further work since it would require much more algebra, even though the final result would be physically more accurate.

2. The Zakharov and Zufiria Hamiltonians

In this section we shall recall some well-known facts about the Hamiltonian formalism of surface water waves (Zakharov [1]). We will study a truncated version of it.

Let $\eta(x)$ be the elevation of the surface wave, $\Phi(x,y)$ the velocity potential and $\psi(x)$ be the velocity potential evaluated at the surface. The dynamical variables are (η,ψ), canonically conjugated. We only consider the case where the bottom is at an infinite depth.

Then the Hamiltonian $H(\eta,\psi)$ is the total energy of the system given as the sum of the kinetic energy H_e and the potential energy H_p, where:

$$H_e = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x \int_{-\infty}^{\eta(x)} \mathrm{d}y \, \left[\left(\partial_x \Phi(x,y)\right)^2 + \left(\partial_y \Phi(x,y)\right)^2\right], \tag{1}$$

and

$$H_p = \frac{1}{2}\int_0^{2\pi} \mathrm{d}x\ \eta^2(x). \tag{2}$$

Let us first introduce the Fourier representation of $\eta(x)$ and $\psi(x)$:

$$\eta(x) = \sum_{k\in\mathbb{Z}} \hat{\eta}_k \mathrm{e}^{ikx} \quad \text{with } \hat{\eta}_k = \hat{\eta}^*_{-k}, \tag{3}$$

$$\psi(x) = \sum_{k\in\mathbb{Z}} \hat{\psi}_k \mathrm{e}^{ikx} \quad \text{with } \hat{\psi}_k = \hat{\psi}^*_{-k} \tag{4}$$

along with the new variables:

$$\hat{a}_k = \left(\hat{\eta}_k|k|^{-\frac{1}{4}} + i\hat{\psi}_k|k|^{\frac{1}{4}}\right)/\sqrt{2}, \tag{5}$$

i.e.

$$\hat{\eta}_k = \left(\hat{a}_k + \hat{a}^*_{-k}\right)|k|^{\frac{1}{4}}/\sqrt{2}, \tag{6}$$

$$\hat{\psi}_k = \left(\hat{a}_k - \hat{a}^*_{-k}\right)|k|^{-\frac{1}{4}}/(i\sqrt{2}). \tag{7}$$

The dynamical variables are now $(\hat{\eta}, \hat{\psi})$, still canonically conjugated, or also $(\hat{a}_k, \hat{a}^*_k)$. The Hamiltonian is then reduced to a system of simple equations:

$$\partial_t \hat{a}_k = -i\frac{\partial H}{\partial \hat{a}^*_k}, \quad k\in\mathbb{Z}. \tag{8}$$

By definition, $\eta(x,t)$ is a travelling wave if and only if there exists a function g and a constant c such that $\eta(x,t) = g(x - c.t)$. This is equivalent to:

$$\partial_t\eta = -c.\partial_x\eta \quad \text{or} \quad \partial_t\hat{\eta}_k = -i.k.c.\hat{\eta}_k \tag{9}$$

or to the requirement that the function $\hat{\eta}_k(t)$ verifies:

$$\forall k,t \quad \mathrm{Log}\left(\hat{\eta}_k(t)\right) = e_k - i.k.c.t \tag{10}$$

for some constants e_k.

The expansion of the Hamiltonian in power series of the variables $\hat{a}$ is formally given by (Krasitskii [23]).

Many models which are derived from the Zakharov Hamiltonian system have been used to study the time evolution of spatially periodic waves, see Stiasnie and Shemer [24], Zufiria [21], and Badulin et al. [25]. In this paper, we choose the 3 waves interaction model introduced by Zufiria, where he assumes that the only non-zero variables are $\hat{a}_1, \hat{a}_2, \hat{a}_3$, the others being frozen to 0. Due to the existence of an extra integral of motion (the total momentum (14), besides the Hamiltonian itself), this is known to be the minimal model that may exhibit a chaotic behavior. Zufiria gives an expansion of the Hamiltonian up to the 4th order of waves amplitudes $\hat{a}$. We can rewrite it as follows:

$$H = \hat{a}_1\hat{a}^*_1 + 2^{\frac{1}{2}}\hat{a}_2\hat{a}^*_2 + 3^{\frac{1}{2}}\hat{a}_3\hat{a}^*_3 + 2^{\frac{1}{2}}\gamma_6\left(\hat{a}_1\hat{a}_2\hat{a}^*_3 + \hat{a}^*_1\hat{a}^*_2\hat{a}_3\right) + 2^{\frac{1}{2}}\gamma_7\left(\hat{a}^2_1\hat{a}^*_2 + \hat{a}^{*2}_1\hat{a}_2\right)$$

$$+\hat a_1\hat a_1^*\left(\frac{1}{8}\hat a_1\hat a_1^* + 2w_1\hat a_2\hat a_2^* + w_2\hat a_3\hat a_3^*\right) + \hat a_2\hat a_2^*\left(\hat a_2\hat a_2^* + 2w_3\hat a_3\hat a_3^*\right) + \frac{27}{8}\left(\hat a_3\hat a_3^*\right)^2$$
$$+2\gamma_8\left(\hat a_1\hat a_2^{*2}\hat a_3 + \hat a_1^*\hat a_2^2\hat a_3^*\right) - \gamma_9\left(\hat a_1^3\hat a_3^* + \hat a_1^{*3}\hat a_3\right), \tag{11}$$

where

$$\gamma_6 = \left(\frac{3}{8}\right)^{\frac{1}{4}}, \quad \gamma_7 = 2^{-\frac{9}{4}}, \quad w_1 = \frac{1}{2} + \frac{\sqrt{2}}{8}, \quad w_2 = \frac{3+\sqrt{3}}{2}, \tag{12}$$

$$w_3 = 3 + \sqrt{\frac{3}{8}}, \quad \gamma_8 = 2^{-\frac{7}{2}}.3^{-\frac{1}{4}}.(2+\sqrt{3}+\sqrt{8}), \quad \gamma_9 = \frac{3^{\frac{1}{4}}}{8}. \tag{13}$$

Zufiria also notices that this Hamiltonian retains all the resonant terms up to order 4, and therefore it can be viewed as the normal form in 3 degrees of freedom.

For the water waves equation, beside the Hamiltonian, there is an other conserved quantity which is the total horizontal momentum given by:

$$\mathbb{I} = \sum_{k=1}^{3} k\ \hat a_k\hat a_k^* \tag{14}$$

(obviously the case $\mathbb{I} = 0$ corresponds to a flat surface, for this Zufiria model, where the sum is only on positives k). Indeed (14) is a conserved quantity as can be checked by derivation and the help of the Hamiltonian equation (8). We can also rederive this fact as follows. Let us introduce the action-angle variables with the following canonical transformation from $(\hat a_1, \hat a_1^*, \hat a_2, \hat a_2^*, \hat a_3, \hat a_3^*)$ to $(A, \theta, \mathbb{I}, \xi, B, \varphi)$ defined by:

$$\hat a_1 = \sqrt{a+A}\ \mathrm{e}^{i(\theta+\xi)}, \tag{15}$$

$$\hat a_2 = \frac{1}{\sqrt{2}}\sqrt{\mathbb{I} - a - 3b - A - 3B}\ \mathrm{e}^{2i\xi}, \tag{16}$$

$$\hat a_3 = \sqrt{b+B}\ \mathrm{e}^{i(\varphi+3\xi)}, \tag{17}$$

with

$$a \geqslant 0, \quad b \geqslant 0, \quad a + 3b \leqslant \mathbb{I} \tag{18}$$

and

$$A \geqslant -a, \ B \geqslant -b, \ A + 3B \leqslant \mathbb{I} - a - 3b. \tag{19}$$

The cases $(a = \mathbb{I}, b = 0)$ or $(a = b = 0)$ or $(a = 0, b = \mathbb{I}/3)$ correspond to the travelling waves of class 1, 2 or 3 (respectively) of the Zufiria classification. By studying the stability of these solutions, he finds that the most important solution, from a dynamical point of view, is the one of class 2. Indeed, for a certain values of the parameter $\mathbb{I}$, the stability of this solution changes.

The change of variables (15)–(17) leads to a new Hamiltonian, still denoted by H:

$$H = c + \omega_1 A + \omega_3 B - \gamma_3 AB - \gamma_4 A^2 - \gamma_5 B^2$$
$$+ 2\left[(a+A)(\mathbb{I} - a - 3b - A - 3B)\right]^{\frac{1}{2}}\left[(b+B)^{\frac{1}{2}}\gamma_6\cos(\theta-\varphi) + (a+A)^{\frac{1}{2}}\gamma_7\cos(2\theta)\right]$$
$$+ 2\left[(a+A)(b+B)\right]^{\frac{1}{2}}\left[(\mathbb{I} - a - 3b - A - 3B)\gamma_8\cos(\theta+\varphi) - (a+A)\gamma_9\cos(3\theta-\varphi)\right], \tag{20}$$

where:

$$c = \gamma_1 + \mathbb{I}\gamma_2 + \frac{\mathbb{I}^2}{4}, \tag{21}$$

$$\gamma_1 = a\tilde{\omega}_1 - b\tilde{\omega}_3 + ab\gamma_3 + a^2\gamma_4 + b^2\gamma_5, \tag{22}$$

$$\gamma_2 = \frac{1}{\sqrt{2}} + a\hat{\omega}_1 + \hat{\omega}_3 b, \tag{23}$$

$$\gamma_3 = \frac{\sqrt{3}}{2}\left(\sqrt{3} - 1 + \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{8}}\right), \tag{24}$$

$$\gamma_4 = \frac{\sqrt{2}+1}{8}, \tag{25}$$

$$\gamma_5 = 3\left(\frac{9}{8} + \sqrt{\frac{3}{8}}\right), \tag{26}$$

$$\omega_1 = 2a\gamma_4 + b\gamma_3 + \hat{\omega}_1\mathbb{I} + \tilde{\omega}_1, \tag{27}$$

$$\omega_3 = a\gamma_3 + 2b\gamma_5 + \hat{\omega}_3\mathbb{I} - \tilde{\omega}_3, \tag{28}$$

$$\hat{\omega}_1 = \frac{1}{4\sqrt{2}}, \qquad \hat{\omega}_3 = \frac{3}{2}\sqrt{\frac{3}{8}}, \tag{29}$$

$$\tilde{\omega}_1 = 1 - \frac{1}{\sqrt{2}}, \qquad \tilde{\omega}_3 = \sqrt{3} + \frac{3}{\sqrt{2}}. \tag{30}$$

From (20) the Hessian of the Hamiltonian is non-degenerate in action variables. This will be used in section 4 in order to apply the KAM method.

Let us notice that H is independent of ξ and hence we see again that $\mathbb{I}$ is a constant of motion.

3. The KAM method

We want a canonical transformation which turns a Hamiltonian written in action-angle variables, into another one for which we can compute an exact trajectory. And so by inversing this transformation we will get an exact trajectory of the initial Hamiltonian system.

Let us consider an Hamiltonian H with L degrees of freedom in action-angle variables:

$$(A, \theta) \in \mathbb{A} \times \mathbb{T}^L, \tag{31}$$

where $\mathbb{T}$ is the torus $\mathbb{R}/2\pi\mathbb{Z}$ and the domain $\mathbb{A}$ is a neighbourhood of a point $A_0 \in \mathbb{R}^L$. A_0 can be taken to be 0 by translation in the action variable.

Assume that H is of class $\mathcal{C}^2$ with respect to A, and let us write $H(A)(\theta)$ instead of $H(A, \theta)$: this means that $H(A)$ is a function of θ. The Taylor expansion of H around $A = 0$ is then:

$$H(A)(\theta) = c + \omega.A - f(\theta) - V(\theta).A - q(A)(\theta), \tag{32}$$

where c is some scalar constant, ω is some vectorial constant $\in \mathbb{R}^L$. Here f, V and $q(A)$ are functions of θ with values respectively in $\mathbb{R}$, $\mathbb{R}^L$, $\mathbb{R}$.

We denote by $'$ the derivative with respect to A, and by $\partial = \partial_\theta$. So that we have $q(0) = q'(0) = 0$ (q is of order 2 in A). We assume that $\mathcal{D}^0[q''(0)]$ is an invertible matrix from $\mathbb{R}^L$ into itself, where $\mathcal{D}^0(f)$ is the average of a function f over the angles, and:

$$\mathcal{D}^* \equiv \mathbf{1} - \mathcal{D}^0 \tag{33}$$

with **1** being the identity matrix.

If $f = V = 0$, the Hamiltonian equations of motions are:

$$\partial_t A = -\partial H = -\partial q(A), \tag{34}$$

$$\partial_t \theta = H' = \omega + q'(A). \tag{35}$$

When A is taken to be zero, we get a trivial equation, and the flow is:

$$\begin{pmatrix} 0 \\ \theta \end{pmatrix} \to \begin{pmatrix} 0 \\ \theta + \omega t \end{pmatrix} \tag{36}$$

for any initial condition θ.

This was the 'integrable case', where $f = V = 0$. Now let us consider the 'quasi-integrable case', i.e. we assume that f and V are small quantities and we define: $\varepsilon = \max(|f|, |V|)$ for some norm. So the Hamiltonian we investigate is quasi-integrable near $A = 0$. To avoid ambiguities we make the hypothesis:

$$\mathcal{D}^0(f) = \mathcal{D}^0(V) = 0. \tag{37}$$

Indeed the quantities $\mathcal{D}^0(f)$ and $\mathcal{D}^0(V)$ can be added to c and ω respectively.

We want a diffeomorphism T from $\mathbb{A} \times \mathbb{T}^L$ to some domain in $\mathbb{R}^L \times \mathbb{T}^L$ which transforms the Hamiltonian H in another one for which we can exhibit some explicit trajectories.

Let us define the translation operator $\mathcal{T}$ acting on the action variable by:

$$\forall A \in \mathbb{R}^L \ \forall Z \in \mathbb{R}^L \quad \big[\mathcal{T}(Z)H\big](A) = H(A + Z) \tag{38}$$

still valid when the members of this equality are functions of θ. Let us also define the dilation operator $\mathbb{D}$ (acting on the action variable) by:

$$\forall A \in \mathbb{R}^L \ \forall \mathcal{C} \in L\big(\mathbb{R}^L, \mathbb{R}^L\big) \quad \big[\mathbb{D}(\mathcal{C})H\big](A) = H(\mathcal{C}A) \tag{39}$$

so that $\mathbb{D}(\mathcal{C})^{-1} = \mathbb{D}(\mathcal{C}^{-1})$.

Let us choose the following ansatz for T:

$$T = T_2.T_1 \quad \text{with} \tag{40}$$

$$T_1 = \mathcal{T}(\dot\alpha - Z), \tag{41}$$

$$T_2 = (1 + G\partial)^{-1}\mathbb{D}(\mathbf{1} + \dot G) \tag{42}$$

and where:

$$Z \in \mathbb{R}^L, \ \alpha : \mathbb{T}^L \to \mathbb{R}, \quad G : \mathbb{T}^L \to \mathbb{R}^L, \quad \dot\alpha \equiv \partial\alpha, \quad \dot G \equiv \partial G \tag{43}$$

so that Z is a constant vector. Let us emphasize that we denote by a dot the derivative with respect to θ. Of course we assume that $\dot G$ is small enough to have $\mathbf{1} + \dot G$ invertible. Notice that T_1, T_2 (and so T) are canonical transformations. Indeed:

$$T_1 A = A + \dot\alpha - Z \implies d(T_1 A) = dA + \ddot\alpha d\theta, \tag{44}$$

$$T_1\theta = \theta \implies d(T_1\theta) \wedge d(T_1 A) = d\theta \wedge (dA + \ddot\alpha d\theta) = d\theta \wedge dA \tag{45}$$

since $d\theta \wedge d\theta = 0$. And instead to prove that T_2 is canonical, let us rather prove this for $T_3 \equiv T_2^{-1} = \mathbb{D}(\mathbf{1} + \dot G)^{-1}(1 + G\partial)$:

$$T_3 A = (\mathbf{1} + \dot G)^{-1} A \implies d(T_3 A) = (\mathbf{1} + \dot G)^{-1} dA + (\cdots) d\theta, \tag{46}$$

$$T_3\theta = (1 + G\partial)\theta = \theta + G \implies d(T_3\theta) = d\theta(\mathbf{1} + \dot G), \tag{47}$$

$$d(T_3\theta) \wedge d(T_3 A) = d\theta(\mathbf{1} + \dot G) \wedge (\mathbf{1} + \dot G)^{-1} dA + d\theta(\mathbf{1} + \dot G) \wedge (\cdots) d\theta = d\theta \wedge dA.$$

Notice also that:

$$T^{-1}\begin{pmatrix} A \\ \theta \end{pmatrix} = \begin{pmatrix} (\mathbf{1} + \dot G)^{-1}(A + Z - \dot\alpha) \\ \theta + G \end{pmatrix}. \tag{48}$$

The transformed Hamiltonian $\hat H = TH$:

$$\hat H(A) = (1 + G\partial)^{-1} H\left[(\mathbf{1} + \dot G)A + \dot\alpha - Z\right] \tag{49}$$

can be written similarly as (32):

$$\hat H(A) = \hat c + \hat\omega.A - \hat f - \hat V.A - \hat q(A), \tag{50}$$

where

$$\hat c = c - \omega.Z - \mathcal{D}^0\left[V.\dot\alpha + q(\dot\alpha - Z)\right], \tag{51}$$

$$\hat\omega = \omega - \mathcal{D}^0\left[V.\dot G + q'(\dot\alpha - Z).(\mathbf{1} + \dot G)\right], \tag{52}$$

$$\hat f = (1 + G\partial)^{-1}\mathcal{D}^*\left[f - \omega.\dot\alpha + V.\dot\alpha - V.Z + q(\dot\alpha - Z)\right], \tag{53}$$

$$\hat V = (1 + G\partial)^{-1}\mathcal{D}^*\left[V.(\mathbf{1} + \dot G) - \omega.\dot G + q'(\dot\alpha - Z).(\mathbf{1} + \dot G)\right] \tag{54}$$

and

$$\hat q(A) = (1 + G\partial)^{-1}\mathcal{D}^*\left[q\left(\dot\alpha - Z + (\mathbf{1} + \dot G).A\right) - q(\dot\alpha - Z) - q'(\dot\alpha - Z).(\mathbf{1} + \dot G).A\right]. \tag{55}$$

Obviously:

$$\hat q(0) = \hat q'(0) = 0. \tag{56}$$

Hence if we manage to find $\alpha,\ G,\ Z$ such that $\hat f = \hat V = 0$ then we will get a trajectory of the transformed Hamiltonian $\hat H$ since the initial data $A = 0$ remains forever frozen to 0 under the evolution of the special Hamiltonian:

$$\hat H(A)(\theta) = \hat c + \hat\omega.A - \hat q(A)(\theta) \tag{57}$$

for which the flow is:

$$\begin{pmatrix} 0 \\ \theta \end{pmatrix} \to \begin{pmatrix} 0 \\ \theta + \hat\omega t \end{pmatrix} \tag{58}$$

for any initial condition θ.

Using the inverse transformation T^{-1}, we get a trajectory of the initial Hamiltonian H:

$$T^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix} \to T^{-1}\begin{pmatrix} 0 \\ \theta + \hat\omega t \end{pmatrix}, \tag{59}$$

where:

$$T^{-1}\begin{pmatrix}0\\ \theta\end{pmatrix}=\begin{pmatrix}(\mathbf{1}+\dot{G})^{-1}(Z-\dot{\alpha})\\ \theta+G\end{pmatrix}. \tag{60}$$

In order to have $\hat{f}=\hat{V}=0$ we require $\alpha,\ G$ and Z to verify:

$$\omega.\dot{\alpha}=\mathcal{D}^{*}\big[f+V.\dot{\alpha}-V.Z+q(\dot{\alpha}-Z)\big], \tag{61}$$

$$\omega.\dot{G}=\mathcal{D}^{*}\big[V.(1+\dot{G})+q'(\dot{\alpha}-Z).(\mathbf{1}+\dot{G})\big]. \tag{62}$$

Let us define the formal inverse of the operator $\omega.\partial$ by:

$$\Gamma\equiv(\omega.\partial)^{-1}.\mathcal{D}^{*} \tag{63}$$

which is defined on any function of the angles, by its action on the exponential function:

$$\forall\nu\in\mathbb{Z}^{L}\setminus\{0\}\quad \Gamma.e_{\nu}\equiv(i.\omega.\nu)^{-1}.e_{\nu},\quad \text{where } e_{\nu}(\theta)=\mathrm{e}^{i\nu.\theta} \tag{64}$$

and $\Gamma.1=0$.

Of course we need to assume an arithmetic condition on ω namely that it satisfies a Diophantine condition:

$$\forall\nu\in\mathbb{Z}^{L}\setminus\{0\}\quad |\omega.\nu|\geqslant\gamma|\nu|^{-\tau} \tag{65}$$

for some numbers $\gamma>0,\ \tau>L-1$. The set of such frequencies has a positive Lebesgue measure (if γ is small enough), as is well known.

Then (61), (62) can be rewritten formally as:

$$\alpha=\Gamma\big[f+V.\dot{\alpha}-V.Z+q(\dot{\alpha}-Z)\big], \tag{66}$$

$$G=\Gamma\big[V.(\mathbf{1}+\dot{G})+q'(\dot{\alpha}-Z).(\mathbf{1}+\dot{G})\big], \tag{67}$$

i.e.:

$$G=\frac{1}{1-\Gamma[V+q'(\dot{\alpha}-Z)]\partial}\ \Gamma\big[V+q'(\dot{\alpha}-Z)\big]. \tag{68}$$

Notice that α is defined by a fixed-point equation (66) $\alpha=\varepsilon Y(\alpha)$. We will adjust the (vector) parameter Z by requiring that $\hat{\omega}=\omega$, i.e.:

$$\mathcal{D}^{0}\big[V.\dot{G}+q'(\dot{\alpha}-Z).(\mathbf{1}+\dot{G})\big]=0. \tag{69}$$

At order ε, the equation for Z is:

$$\mathcal{D}^{0}\big[q''(0)(\dot{\alpha}-Z)\big]=0 \tag{70}$$

and so:

$$Z=\big[\mathcal{D}^{0}\big[q''(0)\big]\big]^{-1}\mathcal{D}^{0}\big[q''(0)\Gamma\dot{f}\big]+\mathrm{O}\big(\varepsilon^{2}\big). \tag{71}$$

To solve (66) and (69), we can use a particular version of KAM method (Vittot [26]; see also Gallavotti [27]).

4. The Zufiria model as a quasi-integrable Hamiltonian

Here we apply the previous KAM method to the Zufiria Hamiltonian (20). We consider $\mathbb{I}$ as a parameter (since $\mathbb{I}$ is constant on any trajectory). So H becomes an Hamiltonian in 2 degrees of freedoms (A,θ,B,φ), i.e. $L=2$. We exclude the cases $\mathbb{I}=0$, or $a=0$, or $b=0$, or $a+3b=\mathbb{I}$. So we can write H under the form (32) with:

$$\begin{aligned} f(\theta,\varphi) = {} & 2\big[a(\mathbb{I}-a-3b)\big]^{\frac{1}{2}}\big[b^{\frac{1}{2}}\gamma_6\cos(\theta-\varphi)+2a^{\frac{1}{2}}\gamma_7\cos(2\theta)\big] \\ & +(ab)^{\frac{1}{2}}\big[(\mathbb{I}-a-3b)\gamma_8\cos(\theta+\varphi)-a\gamma_9\cos(3\theta-\varphi)\big]. \end{aligned} \tag{72}$$

Likewise $V=V=(V_1,V_3)$ with:

$$\begin{aligned} V_1(\theta,\varphi) = {} & [b/a]^{\frac{1}{2}}\frac{(\mathbb{I}-2a-3b)}{[\mathbb{I}-a-3b]^{\frac{1}{2}}}\gamma_6\cos(\theta-\varphi) + [b/a]^{\frac{1}{2}}(\mathbb{I}-3a-3b)\gamma_8\cos(\theta+\varphi) \\ & -3(ab)^{\frac{1}{2}}\gamma_9\cos(3\theta-\varphi)+\frac{(2\mathbb{I}-3a-6b)}{[\mathbb{I}-a-3b]^{\frac{1}{2}}}\gamma_7\cos(2\theta) \end{aligned} \tag{73}$$

and:

$$\begin{aligned} V_3(\theta,\varphi) = {} & [a/b]^{\frac{1}{2}}\frac{(\mathbb{I}-a-6b)}{[\mathbb{I}-a-3b]^{\frac{1}{2}}}\gamma_6\cos(\theta-\varphi) + [a/b]^{\frac{1}{2}}(\mathbb{I}-a-9b)\gamma_8\cos(\theta+\varphi) \\ & -a[a/b]^{\frac{1}{2}}\gamma_9\cos(3\theta-\varphi) - \frac{3a}{[\mathbb{I}-a-3b]^{\frac{1}{2}}}\gamma_7\cos(2\theta). \end{aligned} \tag{74}$$

We choose a and b in the domain $\mathbb{B}$ defined by:

$$\mathbb{B}=\big\{(a,b)\in\mathbb{R}_+^{*2};\ a+3b<\mathbb{I};\ |f| \text{ and } |V|<\varepsilon_0\big\} \tag{75}$$

for the norm:

$$|f|=\sup_{\nu\in\mathbb{Z}^2}\big|\hat{f}(\nu)\big|.F_\sigma(R.\nu) \tag{76}$$

with

$$F_\sigma(\nu)=\sum_{j=0}^{\sigma}\frac{|\nu|^j}{j!} \tag{77}$$

using the Fourier coefficients $\hat{f}$ of f. Likewise $|V|=|V_1|+|V_3|$.

The parameters σ, R, ε_0 can be taken to be $\sigma=3$ (since we have 2 degrees of freedom), $R=1/3$ and $\varepsilon_0=1/10$. $\mathbb{B}$ is non-empty when $\mathbb{I}\leqslant 1/40$. The shape of this domain is shown in *figure 1*.

The generators α, $G=(G_1,G_3)$, $Z=(Z_1,Z_3)$ are given by (66), (68), (69). The solution (59), (60) is written as:

$$T^{-1}\begin{pmatrix}0\\0\\\theta\\\varphi\end{pmatrix}=\begin{pmatrix}(\mathbf{1}+\dot{G})^{-1}\begin{pmatrix}Z_1-\dot{\alpha}_1\\Z_3-\dot{\alpha}_3\end{pmatrix}\\\theta+G_1\\\varphi+G_3\end{pmatrix} \tag{78}$$

with $\dot{\alpha}_1=\partial_\theta\alpha$, $\dot{\alpha}_3=\partial_\varphi\alpha$. Let us recall that the arguments of the functions $\alpha_1,\alpha_3,G_1,G_3$ are θ,φ. Then the flow acts on the angle variables by (59):

$$\theta(t)=\theta(0)+\omega_1 t, \tag{79}$$

$$\varphi(t)=\varphi(0)+\omega_3 t \tag{80}$$

since $\hat{\omega}=\omega$ by the choice of Z. To the first order in ε, we have (cf (66), (68)):

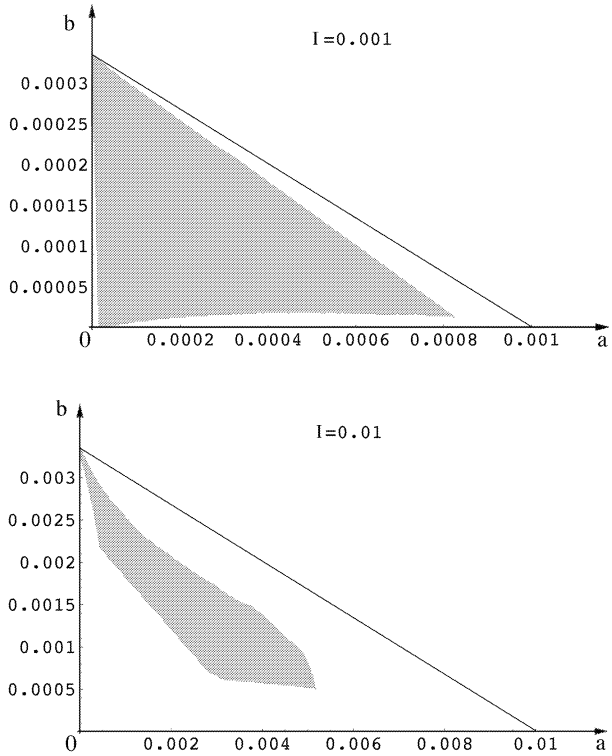

Figure 1. Domain $\mathbb{B}$ (shaded) when $\mathbb{I} = 0.001$ and $\mathbb{I} = 0.01$.

$$\alpha = \Gamma f, \tag{81}$$

$$G_1 = \Gamma(V_1 + \Gamma \partial_\theta f), \tag{82}$$

$$G_3 = \Gamma(V_3 + \Gamma \partial_\varphi f). \tag{83}$$

Now we define the Cantor set $\mathbb{K}$ of Diophantine points (a, b) by the conditions (65) where ω is actually an affine function of a and b ($\mathbb{I}$ being fixed): cf. (27), (28). It is non-empty and has a positive Lebesgue measure (if γ is small enough), as is well known. Let us also define $\widetilde{\mathbb{B}} = \mathbb{B} \cap \mathbb{K}$. We can also prove that it is non-empty and has a positive Lebesgue measure (since $\mathbb{B}$ is an open set).

For any value of $(a, b) \in \widetilde{\mathbb{B}}$ we can write a trajectory of the Hamiltonian (20) under the form (79)–(83) and (71). And so we get a trajectory of the Hamiltonian (11):

$$\hat{a}_1(t) = \sqrt{a + Z_1 - \Gamma \partial_\theta f + \mathrm{O}(\varepsilon^2)}\, \exp i\left(\theta + \Gamma V_1 + \Gamma^2 \partial_\theta f + \xi + \mathrm{O}(\varepsilon^2)\right), \tag{84}$$

$$\hat{a}_3(t) = \sqrt{b + Z_3 - \Gamma \partial_\varphi f + \mathrm{O}(\varepsilon^2)}\, \exp i\left(\varphi + \Gamma V_3 + \Gamma^2 \partial_\varphi f + 3\xi + \mathrm{O}(\varepsilon^2)\right), \tag{85}$$

$$\hat{a}_2(t) = \frac{1}{\sqrt{2}} \sqrt{\mathbb{I} - \hat{a}_1 \hat{a}_1^* - 3\hat{a}_3 \hat{a}_3^*}\ \exp(2i\xi) \tag{86}$$

with

$$\partial_t \xi = \frac{\partial H}{\partial \mathbb{I}} \quad \left(\text{i.e. } \xi = \frac{\mathbb{I}}{2} t + \xi_0 + \mathrm{O}(\varepsilon^2)\right). \tag{87}$$

We have seen in (10) that $\eta(x,t)$ is a travelling wave if and only if:

$$\forall k, t \quad \mathrm{Log}\left(\hat{\eta}_k(t)\right) = e_k - i.k.c.t \tag{88}$$

for some constants e_k, which is equivalent, due to (5), to:

$$\forall k, t \quad \mathrm{Log}\left(\hat{a}_k(t)\right) = \hat{e}_k - i.k.c.t \tag{89}$$

for some other constants $\hat{e}_k$.

In particular taking $k=1$, in order to get a travelling wave, the following should be true:

$$\frac{1}{2}\mathrm{Log}\,(a + Z_1 - \Gamma\partial_\theta f) + i(\theta + \Gamma V_1 + \Gamma^2\partial_\theta f + \xi) = \hat{e}_1 - ict. \tag{90}$$

But this cannot be true since the left-hand side of this formula is not affine in t because f and V are quasi-periodic (cf. (72)–(74)). Hence our solutions are periodic in x, quasi-periodic in t, and not travelling waves (since the Diophantine vector ω is not rational).

5. Conclusion

In this paper we have considered the problem of existence of solutions of a free surface in gravity waves, quasi-periodic in time and periodic in space. We applied the KAM method to the particular Zufiria model of 3 waves interaction which is a truncation of the Zakharov Hamiltonian. From the expressions (84)–(86) we remark that these solutions are not the ones obtained by the Hopf bifurcation, as in Van der Meer [28]. For the Hopf bifurcation, the solution is obtained by a continuous variation of the parameter, which is the celerity, of a permanent form solution. This is not the case for the solutions given by the KAM method, they exist in a domain of the phase space which contains the non-empty Cantor set $\widetilde{\mathbb{B}}$. Moreover, these solutions are not null solutions since α and G are not identically zero. And neither are they permanent form solutions because α and G are not constant functions.

This method may be extended to a finite trucation of the Zakharov Hamiltonian with more modes involved.

In extension to this work, we would like to give a better threshold for the domain of the existence of the solution. Our next aim, unsolved for the moment, would be the case of a system of infinitely many degrees of freedom. It is known to be a difficult problem in KAM theory.

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